Chapter 3 (Solutions)

How are models set up?
II. An introduction to differential equations

3.1 The differential equations are as follows:

$$\frac{dS(t)}{dt} = -\lambda(t)S(t) - vS(t)$$

$$\frac{dS_p(t)}{dt} = -\lambda(t)S_p(t) + w_v V(t) + w_n R(t)$$

$$\frac{dI(t)}{dt} = \lambda(t)S(t) - rI(t)$$

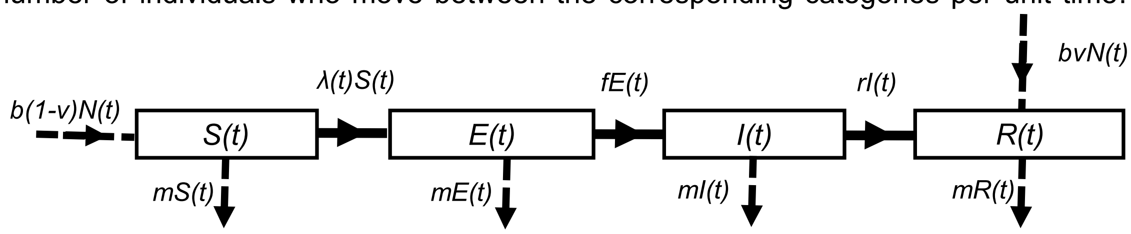
$$\frac{dI_p(t)}{dt} = \lambda(t)S_p(t) - rI_p(t)$$

$$\frac{dV(t)}{dt} = vS(t) - w_v V(t)$$

$$\frac{dR(t)}{dt} = rI(t) - w_n R(t) + rI_p(t)$$

b) The authors would have chosen to use this model rather than an SIRS model so that they could allow the duration of immunity and the infectiousness of infectious persons to depend on whether or not individuals have been infected naturally or vaccinated. The model structure used also allows the authors to allow the susceptibility to infection to differ between those who have been vaccinated and those who have been neither vaccinated nor infected. However, a drawback of having such a high level of detail in the model is that not all of the input parameters that are needed may be known.

3.2 a) The model diagram is as follows; the expressions next to the arrows reflect the number of individuals who move between the corresponding categories per unit time:



b) i) m is interpretable as the *per capita* mortality rate, which is assumed to be identical for all individuals.

ii) $N(t)$ is the total population size at time t.

c) Since both vaccinated individuals and those who are immune because of natural infection have been put into the same compartment, the model assumes that natural infection and vaccination provide the same level of protection..

3.3 a) The differential equations are as follows:

$$\frac{ds(t)}{dt} = -\lambda s(t)$$
$$\frac{dz(t)}{dt} = \lambda s(t)$$

Notice that we have used the symbol λ for the force of infection in these equations, rather than $\lambda(t)$. This reflects the fact that the force of infection is assumed to be the same over time.

b) i) The equations are: $s(t)=e^{-\lambda t}$ or $s_t=(1-\lambda_r)^t$ where λ_r is the risk of infection in each year of life.

ii) Assuming that the proportion (ever) infection is just 1-proportion susceptible, then the proportion ever infected is given by the following:

$$z(t) = 1- e^{-\lambda t}\ , \quad \text{or}$$
$$z_t = 1-(1-\lambda_r)^t$$

c) The following table provides the corresponding values for the proportion ever infected by different ages:

Age (years)	**Force of infection (% per year)**		
	1%	**10%**	**20%**
5	0.0488	0.3935	0.6321
10	0.0952	0.6321	0.8647
20	0.1813	0.8647	0.9817
60	0.4512	0.9975	1.0000

Almost all individuals are predicted to have been infected by age 20 years in the high transmission setting. Since rubella is an immunizing infection, i.e. once infected, individuals are immune for life, very few individuals are infected as adults in high transmission settings. The burden of rubella among adults is therefore likely to be smallest in the high transmission setting. These issues are discussed further in chapter 5.