Chapter 7 (Solutions)

How do models deal with contact patterns?

7.1 Using a similar approach to that used in Panel 7.1, we obtain the following two expressions for the number of new infections among adults which are attributable to contact with children:

$$\beta_{oy} S_o(t) I_y(t) \qquad \text{S7.1}$$

$$\lambda_{oy}(t) S_o(t) \qquad \text{S7.2}$$

Equating expressions S7.1 and S7.2, we obtain the following:

$$\lambda_{oy}(t) S_o(t) = \beta_{oy} S_o(t) I_y(t)$$

Cancelling $S_o(t)$ from both sides of this equation, we obtain the following equation:

$$\lambda_{oy}(t) = \beta_{oy} I_y(t) \qquad \text{S7.3}$$

Similarly, we can obtain the following two expressions for the number of new infections among adults that are attributable to contact with other adults:

$$\beta_{oo} S_o(t) I_o(t) \qquad \text{S7.4}$$

$$\lambda_{oo}(t) S_o(t) \qquad \text{S7.5}$$

Equating expressions S7.4 and S7.5 and cancelling $S_o(t)$ from the resulting equation, we obtain the following equation:

$$\lambda_{oo}(t) = \beta_{oo} I_o(t) \qquad \text{S7.6}$$

Substituting our expressions for $\lambda_{oy}(t) = \beta_{oy} I_y(t)$ and $\lambda_{oo}(t) = \beta_{oo} I_o(t)$ into equation 7.3 in the book, we obtain our intended result, i.e.

$$\overline{\lambda_o(t)} = \beta_{oy} I_y(t) + \beta_{oo} I_o(t)$$

7.2 We will use the notation and definitions for the symbols provided on pages 183 and 184, and set $\beta_{yy}=2\times10^{-4}$ per day, $\beta_{yo}=8\times10^{-4}$ per day, $\beta_{oy}=3\times10^{-4}$ per day, and $\beta_{oo}=7\times10^{-5}$ per day. The answers to the questions are as follows:

a) i) $\lambda_{yy}(t) = \beta_{yy} I_y(t) = 2 \times 10^{-4} \times 20 = 0.004$ per day.

ii) $\lambda_{yo}(t) = \beta_{yo} I_o(t) = 8 \times 10^{-4} \times 50 = 0.04$ per day.

iii) $\overline{\lambda_y(t)} = \lambda_{yy}(t) + \lambda_{yo}(t) = 0.004 + 0.04 = 0.044$ per day.

b) i) $\lambda_{oy}(t) = \beta_{oy} I_y(t) = 3 \times 10^{-4} \times 20 = 0.006$ per day.

ii) $\lambda_{oo}(t) = \beta_{oo} I_o(t) = 7 \times 10^{-5} \times 50 = 0.035$ per day.

iii) $\overline{\lambda_o(t)} = \lambda_{oy}(t) + \lambda_{oo}(t) = 0.006 + 0.035 = 0.095$ per day.

7.3 Using WAIFW matrix $\begin{pmatrix} \beta_1 & 0.5\beta_2 \\ 0.5\beta_2 & \beta_2 \end{pmatrix}$, and assuming that the average force of infection among children and adults is 13% and 4% per year respectively, and that the average numbers of infectious children and adults are 18,956 and 2,859 respectively, then we would need to solve the following matrix equation to obtain values for β_1 and β_2:

$$\begin{pmatrix} \beta_1 & 0.5\beta_2 \\ 0.5\beta_2 & \beta_2 \end{pmatrix} \begin{pmatrix} 18{,}956 \\ 2{,}859 \end{pmatrix} = \begin{pmatrix} 0.13 \\ 0.04 \end{pmatrix}$$

This equation can be written out in full as follows:

$$18{,}956\beta_1 + 2{,}859 \times 0.5\beta_2 = 0.13 \qquad \text{S7.7}$$

$$18{,}956 \times 0.5\beta_2 + 2{,}859 \times \beta_2 = 0.04 \qquad \text{S7.8}$$

Equation S7.8 simplifies to the following:

$$12{,}237\beta_2 = 0.04 \text{ per year.}$$

Dividing both side of this equation by 12,237 leads to $\beta_2 = 3.24 \times 10^{-6}$ per year.

Substituting this value for β_2 into equation S7.7 leads to the following:

$$18{,}956\beta_1 + 2{,}859 \times 0.5 \times 3.24 \times 10^{-6} = 0.13$$

This equation can be rearranged to give the following:

$$18{,}956\beta_1 = 0.12536 \qquad \text{S7.9}$$

Dividing both sides of this equation by 18,956, we obtain $\beta_2 = 6.61 \times 10^{-6}$ per year. Dividing the values obtained for β_1 and β_2 by 365 to obtain values in units of per day leads to the following values: β_1 = 1.81×10^{-8} per day and β_2 = 8.88×10^{-9} per day.

Substituting these values for β_1 and β_2 into the matrix $\begin{pmatrix} \beta_1 & 0.5\beta_2 \\ 0.5\beta_2 & \beta_2 \end{pmatrix}$ leads to the following matrix $\begin{pmatrix} 1.81 \times 10^{-8} & 4.44 \times 10^{-9} \\ 4.44 \times 10^{-9} & 8.88 \times 10^{-9} \end{pmatrix}$ in units of per day. This matrix is identical to matrix R2 that is presented in section 7.4.2.1.1.

7.4 a) We will use the equation $I_i=\lambda_i S_i D$, to calculate the number of infectious persons in age group i, where λ_i and S_i are the force of infection and the susceptible infectious persons in age group i respectively, and D is the duration of infectiousness (=7 days or 7/365 years). The age groups 0-1, 2-4, 5-9, 10-14 and 15-74 years will be denoted using the subscripts 1, 2, 3, 4 and 5 respectively.

The following table summarizes the average numbers of infectious individuals calculated for each age group:

Age group (years)	**0-1**	**2-4**	**5-9**	**10-14**	**15-74**
Average prevaccination force of infection (%/year)	7.7	23.7	51.7	25.5	9.9
Average number susceptible (prevaccination)	1,062,861	1,216,541	493,355	59,269	59,182
Average number of infectious persons	I_1=1,570 (=0.077 × 1,062,861 × 7/365)	I_2=5,529 (=0.237 × 1,216,541 × 7/365)	I_3=4,892 (=0.517 × 493,355 × 7/365)	I_4=290 (=0.255 × 59,269 × 7/365)	I_5=112 (=0.099 × 59,182 × 7/365)

Note that when applying the equation $I_i=\lambda_i S_i D$, the units for the force of infection in age group i and the duration of infectiousness must be consistent: since we used an annual force of infection, the duration of infectiousness used in the equation is also in units of years.

b) i) After some calculations (see below) and assuming that α=1, we obtain the following values for the β parameters for matrix 2:

$\beta_1 = 6.20\times 10^{-6}$ per year

$\beta_2 = 2.11\times 10^{-5}$ per year

$\beta_3 = 7.84\times 10^{-5}$ per year

$\beta_4 = 2.14\times 10^{-5}$ per year

$\beta_5 = 7.99\times 10^{-6}$ per year

The WAIFW matrix is therefore as follows (where the parameters are in units of per year):

$$\begin{pmatrix} 6.20\times10^{-6} & 6.20\times10^{-6} & 6.20\times10^{-6} & 6.20\times10^{-6} & 7.99\times10^{-6} \\ 6.20\times10^{-6} & 2.11\times10^{-5} & 2.11\times10^{-5} & 2.11\times10^{-5} & 7.99\times10^{-6} \\ 6.20\times10^{-6} & 2.11\times10^{-5} & 7.84\times10^{-5} & 2.14\times10^{-5} & 7.99\times10^{-6} \\ 6.20\times10^{-6} & 2.11\times10^{-5} & 2.14\times10^{-5} & 7.84\times10^{-5}{}_3 & 7.99\times10^{-6} \\ 7.99\times10^{-6} & 7.99\times10^{-6} & 7.99\times10^{-6} & 7.99\times10^{-6} & 7.99\times10^{-6} \end{pmatrix}$$

The equation that we need to solve to obtain these values for the β parameters is as follows:

$$\begin{pmatrix} \beta_1 & \beta_1 & \beta_1 & \beta_1 & \beta_5 \\ \beta_1 & \beta_2 & \beta_2 & \beta_2 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & \beta_4 & \alpha\beta_3 & \beta_5 \\ \beta_5 & \beta_5 & \beta_5 & \beta_5 & \beta_5 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix}$$

These equations can be rewritten as follows:

$$\beta_1(I_1 + I_2 + I_3 + I_4) + \beta_5 I_5 = \lambda_1 \qquad \text{S7.10}$$

$$\beta_1 I_1 + \beta_2(I_2 + I_3 + I_4) + \beta_5 I_5 = \lambda_2 \qquad \text{S7.11}$$

$$\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \beta_4 I_4 + \beta_5 I_5 = \lambda_3 \qquad \text{S7.12}$$

$$\beta_1 I_1 + \beta_2 I_2 + \beta_4 I_3 + \alpha\beta_3 I_4 + \beta_5 I_5 = \lambda_4 \qquad \text{S7.13}$$

$$\beta_5(I_1 + I_2 + I_3 + I_4 + I_5) = \lambda_5 \qquad \text{S7.14}$$

Equation S7.14 can be rearranged to give the following:

$$\beta_5 = \frac{\lambda_5}{I_1 + I_2 + I_3 + I_4 + I_5}$$

Substituting for λ_5=0.099 per year and for $I_1 + I_2 + I_3 + I_4 + I_5 = 12{,}393$ into this equation, we obtain the following value for β_5:

$$\beta_5 = 0.099/12{,}393 = 7.99 \times 10^{-6} \text{ per year}$$

Equation S7.10 can be rearranged to give the following expression for β_1:

$$\beta_1 = \frac{\lambda_1 - \beta_5 I_5}{I_1 + I_2 + I_3 + I_4}$$

Substituting for $\beta_5 = 7.99 \times 10^{-6}$ per year, λ_1=0.237 per year and for the corresponding numbers of infectious persons into this equation, we obtain $\beta_1 = 6.20 \times 10^{-6}$ per year.

Equation S7.11 can be rearranged to give the following expression for β_2:

$$\beta_2 = \frac{\lambda_2 - \beta_1 I_1 - \beta_5 I_5}{I_2 + I_3 + I_4}$$

Substituting for $\beta_5 = 7.99 \times 10^{-6}$ per year, $\beta_1 = 6.20 \times 10^{-6}$ per year and for the corresponding numbers of infectious persons into this equation, we obtain $\beta_2 = 2.11 \times 10^{-5}$ per year.

To obtain β_3 and β_4, we can solve equations S7.12 and S7.13 simultaneously. To simplify the notation, we will re-express these two equations as follows:

$$\beta_3 I_3 + \beta_4 I_4 = P \qquad \text{S7.12'}$$

$$\alpha\beta_3 I_4 + \beta_4 I_3 = Q \qquad \text{S7.13'}$$

where $P = \lambda_3 - \beta_1 I_1 - \beta_2 I_2 - \beta_5 I_5$ and $Q = \lambda_4 - \beta_1 I_1 - \beta_2 I_2 - \beta_5 I_5$. Substituting the corresponding values for the force of infection, the β values and the numbers of infectious persons into these equations, we see that P=0.3895 and Q=0.1265 (to 4 decimal places).

Multiplying equations S7.12' and S7.13' by I_3 and I_4 respectively, we obtain the following:

$$\beta_3 I_3^2 + \beta_4 I_4 I_3 = P I_3 \qquad \text{S7.15}$$

$$\alpha\beta_3 I_4^2 + \beta_4 I_3 I_4 = Q I_4 \qquad \text{S7.16}$$

Subtracting equation S7.16 from equation S7.15, we obtain the following equation.

$$\beta_3 (I_3^2 - \alpha I_4^2) = P I_3 - Q I_4$$

After rearranging, we obtain the following expression for β_3:

$$\beta_3 = \frac{P I_3 - Q I_4}{I_3^2 - \alpha I_4^2}$$

Substituting for *P, Q,* I_3 and I_4 into this equation leads to the result that $\beta_3 = 7.84 \times 10^{-5}$ per year.

Rearranging equation S7.12', we obtain the following expression for β_4:

$$\beta_4 = \frac{P - \beta_3 I_3}{I_4}$$

Substituting for *P*, I_3, I_4 and β_3 into this equation, we obtain $\beta_4 = 2.14 \times 10^{-5}$ per year.

ii) To calculate R_0, we first need to calculate the Next Generation Matrix, using the number of infectious individuals among those in age group *i* resulting from individuals in age group *j*, as obtained using the expression $N_i \beta_{ij} D$. Here, N_i is the number of individuals in age group *i*, and β_{ij} is the rate at which specific susceptible individuals in age group *i* come into effective contact with specific infectious individuals in age group *j* and *D* (=7 days) is the duration of infectiousness.

N_i is given by the width of age group *i* multiplied by 650,000 (the number of individuals in each single year age category), as follows:

Age group (years)	**0-1***	**2-4**	**5-9**	**10-14**	**15-74**
N_i	1,137,500 (= 1.75×650,000)	1,950,000 (= 3×650,000)	3,250,000 (= 5×650,000)	3,250,000 (= 5×650,000)	39,000,000 (= 60×650,000)

* Note that calculations for 0-1 year olds assume that individuals have maternal immunity for the first 3 months of life.

The Next Generation Matrix should resemble the following:

$$
\begin{array}{c c}
\textbf{Age grp (yrs)} & \begin{array}{ccccc} \text{0-1} & \text{2-4} & \text{5-9} & \text{10-14} & \text{15-74} \end{array} \\
\begin{array}{c} \text{0-1} \\ \text{2-4} \\ \text{5-9} \\ \text{10-14} \\ \text{15-74} \end{array} &
\begin{pmatrix}
0.135 & 0.135 & 0.135 & 0.135 & 0.174 \\
0.232 & 0.790 & 0.790 & 0.79 & 0.299 \\
0.386 & 1.317 & 4.884 & 1.335 & 0.498 \\
0.386 & 1.317 & 1.335 & 4.884 & 0.498 \\
5.975 & 5.975 & 5.975 & 5.975 & 5.975
\end{pmatrix}
\end{array}
$$

Adapting the model files provided for calculating R_0, we obtain a value for R_0 of 9.1.

iii) The herd immunity threshold for this matrix is $1\text{-}1/R_0$ or $100\times(1\text{-}1/9.1)\approx$ 89%

c) Increasing the size of α increases the amount of contact between 10-14 year olds and leads to an increase in the size of R_0 and the herd immunity threshold. For example, if α=2, R_0 equals 11.4 and the herd immunity threshold is about 91%. Consequently, the greater the amount of contact between 10-14 year olds, the more difficult it is to control transmission through vaccination.

d) The following summarizes the values for R_n that are obtained for different values for α:

α	**1**	**1.25**	**1.5**	**1.75**	**2**
R_n	0.96	0.99	1.04	1.11	1.2

In general, the greater the value for α, the greater the value for R_n. The values obtained for R_n are generally consistent with those in Figure 7.17.

7.5 a) The Next Generation Matrix is given by the following: $\begin{pmatrix} 1.784 & 0.188 \\ 0.383 & 0.766 \end{pmatrix}$, which results in a value for R_0 of 1.85.

b) To answer this question, we calculate the reproduction number using the following Next Generation Matrix

$$\begin{pmatrix} \beta_{yy} S_y D & \beta_{yo} S_y D \\ \beta_{oy} S_o D & \beta_{oo} S_o D \end{pmatrix}$$

where S_y and S_o are the numbers of susceptible children and adults (defined as those aged <15 and ≥15 years respectively), calculated after incorporating the appropriate vaccination coverage for each vaccination scenario. The β parameters are as defined in the text, and D is the duration of infectiousness (2 days):

The following table summarizes the number of susceptible individuals for each vaccination scenario, the Next Generation Matrix and the values for the reproduction number:

Individuals targeted	**Number of susceptible** Children (S_y)	Adults (S_o)	**Next Generation Matrix**	**Reproduction number**
No vaccination	2639	5361	$\begin{pmatrix} 1.784 & 0.188 \\ 0.383 & 0.766 \end{pmatrix}$	1.85 (=R_0)
Children only	139 (=2639-2500)	5361 (=5361-0)	$\begin{pmatrix} 0.094 & 0.010 \\ 0.383 & 0.766 \end{pmatrix}$	0.77
Adults only	2639 (=2639-0)	2861 (=5361-2500)	$\begin{pmatrix} 1.784 & 0.188 \\ 0.204 & 0.409 \end{pmatrix}$	1.81
Same proportion of children and adults*	1814 (=2639×(1-0.3125))	3686 (=5361×(1-0.3125))	$\begin{pmatrix} 1.226 & 0.130 \\ 0.263 & 0.526 \end{pmatrix}$	1.27
Equal numbers of children and adults	1389 (=2639-1250)	4111 (=5361-1250)	$\begin{pmatrix} 0.939 & 0.099 \\ 0.294 & 0.587 \end{pmatrix}$	1.01

*The proportion of children and adults that need to be targeted with this strategy equals the number of vaccine doses available ÷ population size = 2500/(2639+5361)=31.25%

The smallest value for the reproduction number is associated with the strategy of vaccinating only children, which suggests that, of the four strategies, this approach may be the best way of distributing the vaccine stocks. However, we would also need to account for the severity of influenza and the mortality rates in other age groups before making the final decision about which vaccination strategy should be adopted.